

# DELTA-DISCRETE $G$ -SPECTRA AND ITERATED HOMOTOPY FIXED POINTS

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ABSTRACT. Let  $G$  be a profinite group with finite virtual cohomological dimension and let  $X$  be a discrete  $G$ -spectrum. If  $H$  and  $K$  are closed subgroups of  $G$ , with  $H \triangleleft K$ , then, in general, the  $K/H$ -spectrum  $X^{hH}$  is not known to be a continuous  $K/H$ -spectrum, so that it is not known (in general) how to define the iterated homotopy fixed point spectrum  $(X^{hH})^{hK/H}$ . To address this situation, we define homotopy fixed points for delta-discrete  $G$ -spectra and show that the setting of delta-discrete  $G$ -spectra gives a good framework within which to work. In particular, we show that by using delta-discrete  $K/H$ -spectra, there is always an iterated homotopy fixed point spectrum, denoted  $(X^{hH})^{h_\delta K/H}$ , and it is just  $X^{hK}$ . Additionally, we show that for any delta-discrete  $G$ -spectrum  $Y$ ,  $(Y^{h_\delta H})^{h_\delta K/H} \simeq Y^{h_\delta K}$ . Furthermore, if  $G$  is an *arbitrary* profinite group, there is a delta-discrete  $G$ -spectrum  $X_\delta$  that is equivalent to  $X$  and, though  $X^{hH}$  is not even known in general to have a  $K/H$ -action, there is always an equivalence  $((X_\delta)^{h_\delta H})^{h_\delta K/H} \simeq (X_\delta)^{h_\delta K}$ . Therefore, delta-discrete  $L$ -spectra, by letting  $L$  equal  $H$ ,  $K$ , and  $K/H$ , give a way of resolving undesired deficiencies in our understanding of homotopy fixed points for discrete  $G$ -spectra.

## 1. INTRODUCTION

Let  $\mathbf{Spt}$  be the stable model category of Bousfield-Friedlander spectra of simplicial sets. Also, given a profinite group  $G$ , let  $\mathbf{Spt}_G$  be the model category of discrete  $G$ -spectra, in which a morphism  $f$  is a weak equivalence (cofibration) if and only if  $f$  is a weak equivalence (cofibration) in  $\mathbf{Spt}$  (see [3, Section 3]). Given a fibrant replacement functor

$$(-)_{fG} : \mathbf{Spt}_G \rightarrow \mathbf{Spt}_G, \quad X \mapsto X_{fG},$$

so that there is a natural trivial cofibration  $X \rightarrow X_{fG}$ , with  $X_{fG}$  fibrant, in  $\mathbf{Spt}_G$ , the  $G$ -homotopy fixed point spectrum  $X^{hG}$  is defined by

$$X^{hG} = (X_{fG})^G.$$

Let  $H$  and  $K$  be closed subgroups of  $G$ , with  $H$  normal in  $K$ , and, as above, let  $X$  be a discrete  $G$ -spectrum. Then  $K/H$  is a profinite group, and it is reasonable to expect that  $X^{hH}$  is some kind of a continuous  $K/H$ -spectrum, so that the iterated homotopy fixed point spectrum  $(X^{hH})^{hK/H}$  can be formed. Additionally, one might expect  $(X^{hH})^{hK/H}$  to just be  $X^{hK}$ : following Dwyer and Wilkerson (see [11, pg. 434]), when these two homotopy fixed point spectra are equivalent to each other, for all  $H$ ,  $K$ , and  $X$ , we say that homotopy fixed points for  $G$  have the *transitivity property*.

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Under hypotheses on  $G$  and  $X$  that are different from those above, there are various cases where the iterated homotopy fixed point spectrum is well-behaved along the lines suggested above. For example, by [11, Lemma 10.5], when  $G$  is any discrete group, with  $N \triangleleft G$  and  $\mathcal{X}$  any  $G$ -space, the iterated homotopy fixed point space  $(\mathcal{X}^{hN})^{hG/N}$  is always defined and is just  $\mathcal{X}^{hG}$ . Similarly, by [28, Theorem 7.2.3], if  $E$  is an  $S$ -module and  $A \rightarrow B$  is a faithful  $E$ -local  $G$ -Galois extension of commutative  $S$ -algebras, where  $G$  is a stably dualizable group, then if  $N$  is an allowable normal subgroup of  $G$ ,  $(B^{hN})^{hG/N}$  is defined and is equivalent to  $B^{hG}$ .

Also, by [27, Theorem 4.6, (4)], if  $G$  is any profinite group and  $P$  is a symmetric profinite  $G$ -spectrum, and if  $\hat{P}^{hG}$  denotes the profinite homotopy fixed point spectrum (see [27, Remark 4.2]), then, if  $N$  is any closed normal subgroup of  $G$ , there is a stable equivalence

$$(\hat{P}^{hN})^{hG/N} \simeq \hat{P}^{hG}$$

of profinite symmetric spectra.

Let  $k$  be a spectrum such that the Bousfield localization  $L_k(-)$  is equivalent to  $L_M L_T(-)$ , where  $M$  is a finite spectrum and  $T$  is smashing, and let  $A$  be a  $k$ -local commutative  $S$ -algebra. If a spectrum  $E$  is a consistent profaithful  $k$ -local profinite  $G$ -Galois extension of  $A$  of finite vcd (the meaning of these terms is explained in [1]), then, by [1, Proposition 7.1.4, Theorem 7.1.6],

$$(E^{h_k N})^{h_k G/N} \simeq E^{h_k G},$$

for any closed normal subgroup  $N$  of  $G$ , where, for example,  $(-)^{h_k G}$  denotes the  $k$ -local homotopy fixed points (as defined in [1, Section 6.1]).

Finally, let  $G$  be any compact Hausdorff group, let  $R$  be an orthogonal ring spectrum satisfying the assumptions of [14, Section 11.1, lines 1–3], and let  $\mathcal{M}_R$  be the category of  $R$ -modules. Also, let  $M$  be any pro- $G$ - $R$ -module (so that, for example, the pro-spectrum  $M$  is a pro- $R$ -module and a pro-orthogonal  $G$ -spectrum). By [14, Proposition 11.5], if  $N$  is any closed normal subgroup of  $G$ , then there is an equivalence

$$(M^{h_G N})^{hG/N} \simeq M^{hG}$$

of pro-spectra in the Postnikov model structure on the category of pro-objects in  $\mathcal{M}_R$ . Here,  $M^{h_G N}$  is the  $N$ - $G$ -homotopy fixed point pro-spectrum of [14, Definition 11.3] and, as discussed in [14, pg. 165], there are cases when  $M^{h_G N} \simeq M^{hN}$ .

The above results show that when one works with the hypotheses of  $G$  is profinite and  $X \in \text{Spt}_G$  for the first time, it certainly is not unreasonable to hope that the expression  $(X^{hH})^{hK/H}$  makes sense and that it fits into an equivalence  $(X^{hH})^{hK/H} \simeq X^{hK}$ . But it turns out that, in this setting, in general, these constructions are more subtle than the above results might suggest. For example, as explained in [4, Section 5],  $X^{hH}$  is not even known to be a  $K/H$ -spectrum. However, when  $G$  has finite virtual cohomological dimension (*finite vcd*; that is, there exists an open subgroup  $U$  and a positive integer  $m$  such that the continuous cohomology  $H_c^s(U; M) = 0$ , whenever  $s > m$  and  $M$  is a discrete  $U$ -module), then [4, Corollary 5.4] shows that  $X^{hH}$  is always weakly equivalent to a  $K/H$ -spectrum.

But, as explained in detail in [5, Section 3], even when  $G$  has finite vcd, it is not known, in general, how to view  $X^{hH}$  as a continuous  $K/H$ -spectrum (in the sense of [3, 1]), so that it is not known how to form  $(X^{hH})^{hK/H}$ . For example, when  $G = K = \mathbb{Z}/p \times \mathbb{Z}/q$ , where  $p$  and  $q$  are distinct primes, and  $H = \mathbb{Z}/p$ , Ben Wieland found an example of a discrete  $G$ -spectrum  $Y$  such that  $Y^{hH}$  is not a discrete

$K/H$ -spectrum (see [5, Appendix A]). More generally, it is not known if  $Y^{hH}$  is a continuous  $K/H$ -spectrum, and there is no known construction of  $(Y^{hH})^{hK/H}$ .

By [5, Section 4], if  $G$  is any profinite group and  $X$  is a hyperfibrant discrete  $G$ -spectrum, then  $X^{hH}$  is always a discrete  $K/H$ -spectrum, and hence,  $(X^{hH})^{hK/H}$  is always defined. However, it is not known if  $(X^{hH})^{hK/H}$  must be equivalent to  $X^{hK}$ . Also, [5, Section 4] shows that if  $X$  is a totally hyperfibrant discrete  $G$ -spectrum, then  $(X^{hH})^{hK/H}$  is  $X^{hK}$ . But, as implied by our remarks above regarding  $Y$ , it is not known that all the objects in  $\text{Spt}_G$  are hyperfibrant, let alone totally hyperfibrant.

The above discussion makes it clear that there are nontrivial gaps in our understanding of iterated homotopy fixed points in the world of  $\text{Spt}_G$ . To address these deficiencies, in this paper we define and study homotopy fixed points for *delta-discrete  $G$ -spectra*,  $(-)^{h_\delta G}$ , and within this framework, when  $G$  has finite vcd and  $X \in \text{Spt}_G$ , we find that the iterated homotopy fixed point spectrum  $(X^{hH})^{h_\delta K/H}$  is always defined and is equivalent to  $X^{hK}$ . In fact, when  $G$  has finite vcd, if  $Y$  is one of these delta-discrete  $G$ -spectra, then there is an equivalence

$$(Y^{h_\delta H})^{h_\delta K/H} \simeq Y^{h_\delta K}.$$

Before introducing this paper's approach to iterated homotopy fixed points in  $\text{Spt}_G$  in more detail, we quickly discuss some situations where the difficulties with iteration that were described earlier vanish, since it is helpful to better understand where the obstacles in "iteration theory" are.

In general, for any profinite group  $G$  and  $X \in \text{Spt}_G$ ,

$$(X^{h\{e\}})^{hK/\{e\}} = ((X_{fG})^{\{e\}})^{hK} \simeq X^{hK}$$

and, if  $H$  is open in  $K$ , then  $X_{fK}$  is fibrant in  $\text{Spt}_H$  and  $(X_{fK})^H$  is fibrant in  $\text{Spt}_{K/H}$ , so that

$$(X^{hH})^{hK/H} = ((X_{fK})^H)^{hK/H} \simeq ((X_{fK})^H)^{K/H} = (X_{fK})^K = X^{hK}$$

(see [1, Proposition 3.3.1] and [5, Theorem 3.4]). Thus, in general, the difficulties in forming the iterated homotopy fixed point spectrum occur only when  $H$  is a nontrivial non-open (closed normal) subgroup of  $K$ .

Now let  $N$  be any nontrivial closed normal subgroup of  $G$ . There are cases where, thanks to a particular property that  $G$  has, the spectrum  $(X^{hN})^{hG/N}$  is defined, with

$$(1.1) \quad (X^{hN})^{hG/N} \simeq X^{hG}.$$

To explain this, we assume that  $G$  is infinite:

- if  $G$  has finite cohomological dimension (that is, there exists a positive integer  $m$  such that  $H_c^s(G; M) = 0$ , whenever  $s > m$  and  $M$  is a discrete  $G$ -module), then, by [1, Corollary 3.5.6],  $X^{hN}$  is a discrete  $G/N$ -spectrum, so that  $(X^{hN})^{hG/N}$  is defined, and, as hoped, the equivalence in (1.1) holds;
- the profinite group  $G$  is *just infinite* if  $N$  always has finite index in  $G$  (for more details about such groups, see [33]; this interesting class of profinite groups includes, for example, the finitely generated pro- $p$  Nottingham group over the finite field  $\mathbb{F}_{p^n}$ , where  $p$  is any prime and  $n \geq 1$ , and the just infinite profinite branch groups (see [17])), and thus, if  $G$  is just infinite, then  $N$  is always open in  $G$ , so that, as explained above, (1.1) is valid; and

- if  $G$  has the property that every nontrivial closed subgroup is open, then, by [26, Corollary 1] (see also [8]),  $G$  is topologically isomorphic to  $\mathbb{Z}_p$ , for some prime  $p$ , and, as before, (1.1) holds.

In addition to the above cases, there is a family of examples in chromatic stable homotopy theory where iteration works in the desired way. Let  $k$  be any finite field containing  $\mathbb{F}_{p^n}$ , where  $p$  is any prime and  $n$  is any positive integer. Given any height  $n$  formal group law  $\Gamma$  over  $k$ , let  $E(k, \Gamma)$  be the Morava  $E$ -theory spectrum that satisfies

$$\pi_*(E(k, \Gamma)) = W(k)[[u_1, \dots, u_{n-1}]] [u^{\pm 1}],$$

where  $W(k)$  denotes the Witt vectors, the degree of  $u$  is  $-2$ , and the complete power series ring  $W(k)[[u_1, \dots, u_{n-1}]] \cdot u^0$  is in degree zero (see [15, Section 7]). Also, let  $G = S_n \rtimes \text{Gal}(k/\mathbb{F}_p)$ , the extended Morava stabilizer group:  $G$  is a compact  $p$ -adic analytic group, and hence, has finite vcd, and, by [15],  $G$  acts on  $E(k, \Gamma)$ . Then, by using [7, 6, 28, 1] and the notion of total hyperfibrancy, [5] shows that

$$(E(k, \Gamma)^{hH})^{hK/H} \simeq E(k, \Gamma)^{hK},$$

for all  $H$  and  $K$  defined as usual. Here,  $E(k, \Gamma)$  is a continuous  $G$ -spectrum and not a discrete  $G$ -spectrum ( $\pi_0(E(k, \Gamma))$  is not a discrete  $G$ -module). If  $F$  is any finite spectrum that is of type  $n$ , then, as in [3, Corollary 6.5],  $E(k, \Gamma) \wedge F$  is a discrete  $G$ -spectrum and, again by the technique of [5],

$$((E(k, \Gamma) \wedge F)^{hH})^{hK/H} \simeq (E(k, \Gamma) \wedge F)^{hK}.$$

We note that when  $E(k, \Gamma) = E_n$ , the Lubin-Tate spectrum, [5, pg. 2883] reviews some examples of how  $(E(k, \Gamma)^{hH})^{hK/H}$  plays a useful role in chromatic homotopy theory.

Now we explain the approach of this paper to iterated homotopy fixed points in more detail. Let  $G$  be an arbitrary profinite group and, as usual, let  $X \in \text{Spt}_G$ . Also, let  $c(\text{Spt}_G)$  be the category of cosimplicial discrete  $G$ -spectra (that is, the category of cosimplicial objects in  $\text{Spt}_G$ ). If  $Z$  is a spectrum (which, in this paper, always means Bousfield-Friedlander spectrum), we let  $Z_{k,l}$  denote the  $l$ -simplices of the  $k$ th simplicial set  $Z_k$  of  $Z$ . Then  $\text{Map}_c(G, X)$  is the discrete  $G$ -spectrum that is defined by

$$\text{Map}_c(G, X)_{k,l} = \text{Map}_c(G, X_{k,l}),$$

the set of continuous functions  $G \rightarrow X_{k,l}$ . The  $G$ -action on  $\text{Map}_c(G, X)$  is given by  $(g \cdot f)(g') = f(g'g)$ , for  $g, g' \in G$  and  $f \in \text{Map}_c(G, X_{k,l})$ . As explained in [3, Definition 7.1], the functor

$$\text{Map}_c(G, -): \text{Spt}_G \rightarrow \text{Spt}_G, \quad X \mapsto \text{Map}_c(G, X),$$

forms a triple and there is a cosimplicial discrete  $G$ -spectrum  $\text{Map}_c(G^\bullet, X)$ , where, for each  $[n] \in \Delta$ , the  $n$ -cosimplices satisfy the isomorphism

$$\text{Map}_c(G^\bullet, X)^n \cong \text{Map}_c(G^{n+1}, X).$$

Following [3, Remark 7.5], let

$$\widehat{X} = \text{colim}_{N \triangleleft_o G} (X^N)_f,$$

a filtered colimit over the open normal subgroups of  $G$ . Here,  $(-)_f: \text{Spt} \rightarrow \text{Spt}$  denotes a fibrant replacement functor for the model category  $\text{Spt}$ . Notice that  $\widehat{X}$

is a discrete  $G$ -spectrum and a fibrant object in  $\text{Spt}$ . Now let

$$X_\delta = \text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X}).$$

(In the subscript of “ $X_\delta$ ,” instead of “ $\Delta$ ,” we use its less obtrusive and lowercase counterpart.) In the definition of  $X_\delta$  (and everywhere else in this paper), the homotopy limit (as written in the definition) is formed in  $\text{Spt}$  (and not in  $\text{Spt}_G$ ). As explained in Section 2, there is a natural  $G$ -equivariant map

$$\Psi: X \xrightarrow{\simeq} X_\delta$$

that is a weak equivalence in  $\text{Spt}$ . Since  $\widehat{X}$  is a fibrant spectrum,  $\text{Map}_c(G^\bullet, \widehat{X})^n$  is a fibrant spectrum (by applying [3, Corollary 3.8, Lemma 3.10]), for each  $[n] \in \Delta$ .

The map  $\Psi$  and the features of its target  $X_\delta$  motivate the following definition.

**Definition 1.2.** Let  $X^\bullet$  be a cosimplicial discrete  $G$ -spectrum such that  $X^n$  is fibrant in  $\text{Spt}$ , for each  $[n] \in \Delta$ . Then the  $G$ -spectrum

$$\text{holim}_\Delta X^\bullet$$

is a *delta-discrete  $G$ -spectrum*.

The weak equivalence  $\Psi$  gives a natural way of associating a delta-discrete  $G$ -spectrum to every discrete  $G$ -spectrum. Also, we will see that  $X_\delta$  plays a useful role in our work on iterated homotopy fixed points.

Now we give several more useful definitions, including a definition of weak equivalence for the setting of delta-discrete  $G$ -spectra.

**Definition 1.3.** Let  $c(\text{Spt})$  be the category of cosimplicial spectra. If  $Z$  is a spectrum, let  $\text{cc}^\bullet(Z)$  denote the constant cosimplicial object (in  $c(\text{Spt})$ ) on  $Z$ . Also, given a discrete  $G$ -spectrum  $X$ , let

$$\text{cc}_G(X) = \text{holim}_\Delta \text{cc}^\bullet(\widehat{X}).$$

Notice that  $\text{cc}_G(X)$  is a delta-discrete  $G$ -spectrum.

**Definition 1.4.** If the morphism  $f^\bullet: X^\bullet \rightarrow Y^\bullet$  of cosimplicial discrete  $G$ -spectra is an *objectwise weak equivalence* (that is,  $f^n: X^n \rightarrow Y^n$  is a weak equivalence in  $\text{Spt}_G$ , for each  $[n] \in \Delta$ ), such that the induced  $G$ -equivariant map

$$f = \text{holim}_\Delta f^\bullet: \text{holim}_\Delta X^\bullet \rightarrow \text{holim}_\Delta Y^\bullet$$

has source and target equal to delta-discrete  $G$ -spectra, then  $f$  is a weak equivalence in  $\text{Spt}$  (since all the  $X^n$  and  $Y^n$  are fibrant spectra). We call such a map  $f$  a *weak equivalence of delta-discrete  $G$ -spectra*.

Now we define the key notion of homotopy fixed points for delta-discrete  $G$ -spectra.

**Definition 1.5.** Given a delta-discrete  $G$ -spectrum  $\text{holim}_\Delta X^\bullet$ , the *homotopy fixed point spectrum*  $(\text{holim}_\Delta X^\bullet)^{h_\delta G}$  is given by

$$(\text{holim}_\Delta X^\bullet)^{h_\delta G} = \text{holim}_{[n] \in \Delta} (X^n)^{hG},$$

where  $(X^n)^{hG}$  is the homotopy fixed point spectrum of the discrete  $G$ -spectrum  $X^n$ . We use the notation “ $(-)^{h_\delta G}$ ” and the phrase “delta-discrete homotopy fixed points” to refer to the general operation of taking the homotopy fixed points of a delta-discrete  $G$ -spectrum.

We show that the homotopy fixed points  $(-)^{h_\delta G}$  for delta-discrete  $G$ -spectra have the following properties:

- (a) when  $G$  is a finite group, the homotopy fixed points of a delta-discrete  $G$ -spectrum agree with the usual notion of homotopy fixed points for a finite group;
- (b) for any profinite group  $G$ , the homotopy fixed points of delta-discrete  $G$ -spectra can be viewed as the total right derived functor of

$$\lim_{\Delta}(-)^G: c(\mathrm{Spt}_G) \rightarrow \mathrm{Spt},$$

where  $c(\mathrm{Spt}_G)$  has the injective model category structure (which is defined in Section 3);

- (c) given  $X \in \mathrm{Spt}_G$  and the delta-discrete  $G$ -spectrum  $X_\delta$  associated to  $X$ , then, if  $G$  has finite vcd, there is a weak equivalence

$$X^{hL} \xrightarrow{\simeq} (X_\delta)^{h_\delta L}$$

in  $\mathrm{Spt}$ , for every closed subgroup  $L$  of  $G$ ;

- (d) more generally, if  $G$  is any profinite group and  $X \in \mathrm{Spt}_G$ , then there is a  $G$ -equivariant map  $X \xrightarrow{\simeq} \mathrm{cc}_G(X)$  that is a weak equivalence of spectra and a weak equivalence

$$X^{hG} \xrightarrow{\simeq} (\mathrm{cc}_G(X))^{h_\delta G};$$

- (e) if  $f$  is a weak equivalence of delta-discrete  $G$ -spectra, then the induced map  $f^{h_\delta G} = \mathrm{holim}_{[n] \in \Delta} (f^n)^{h_\delta G}$  is a weak equivalence in  $\mathrm{Spt}$  (since each  $(f^n)^{h_\delta G}$  is a weak equivalence between fibrant spectra); and
- (f) if  $G$  has finite vcd, with closed subgroups  $H$  and  $K$  such that  $H \triangleleft K$ , and if  $Y$  is a delta-discrete  $G$ -spectrum, then  $Y^{h_\delta H}$  is a delta-discrete  $K/H$ -spectrum, and

$$(Y^{h_\delta H})^{h_\delta K/H} \simeq Y^{h_\delta K},$$

so that  $G$ -homotopy fixed points of delta-discrete  $G$ -spectra have the transitivity property.

In the above list of properties, (a) is Theorem 4.2, (b) is justified in Theorem 3.4, (c) is Lemma 2.7, (d) is verified right after the proof of Theorem 4.2, and (f) is obtained in Theorem 5.4 (and the three paragraphs that precede it).

Notice that property (b) above shows that the *homotopy* fixed points of delta-discrete  $G$ -spectra are the total right derived functor of fixed points, in the appropriate sense. Also, (d) shows that, for any  $G$  and any  $X \in \mathrm{Spt}_G$ , the delta-discrete  $G$ -spectrum  $\mathrm{cc}_G(X)$  is equivalent to  $X$  and their homotopy fixed points are the same. Thus, the setting of delta-discrete  $G$ -spectra and the homotopy fixed points  $(-)^{h_\delta G}$  includes and generalizes the category of discrete  $G$ -spectra and the homotopy fixed points  $(-)^{hG}$ . Therefore, properties (a) – (f) show that the homotopy fixed points of a delta-discrete  $G$ -spectrum are a good notion that does indeed deserve to be called “homotopy fixed points.”

Now suppose that  $G$  has finite vcd and, as usual, let  $X \in \mathrm{Spt}_G$ . As above, let  $H$  and  $K$  be closed subgroups of  $G$ , with  $H$  normal in  $K$ . In Lemma 2.8, we show that, by making a canonical identification,  $X^{hH}$  is a delta-discrete  $K/H$ -spectrum. Thus, it is natural to form the iterated homotopy fixed point spectrum  $(X^{hH})^{h_\delta K/H}$  and, by Theorem 2.9, there is a weak equivalence

$$X^{hK} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

In this way, we show that when  $G$  has finite vcd, by using delta-discrete  $K/H$ -spectra, there is a sense in which the iterated homotopy fixed point spectrum can always be formed and the transitivity property holds.

More generally, in Theorem 6.3, we show that for *any*  $G$ , though it is not known if  $X^{hH}$  always has a  $K/H$ -action (as mentioned earlier), there is an equivalence

$$(1.6) \quad ((X_\delta)^{h_\delta H})^{h_\delta K/H} \simeq (X_\delta)^{h_\delta K}.$$

Thus, for any  $G$ , by using  $(-)_\delta$  and  $(-)^{h_\delta H}$ , the delta-discrete homotopy fixed points for discrete  $G$ -spectra are – *in general* – transitive. Also, there is a map

$$\rho(X)_H : X^{hH} \rightarrow (X_\delta)^{h_\delta H}$$

that relates the “discrete homotopy fixed points”  $X^{hH}$  to  $(X_\delta)^{h_\delta H}$ , and this map is a weak equivalence whenever the map  $\operatorname{colim}_{U \triangleleft_o H} \Psi^U$  is a weak equivalence (see Theorem 6.2 and the discussion that precedes it).

For any  $G$  and  $X$ , since the  $G$ -equivariant map  $\Psi : X \rightarrow X_\delta$  is a weak equivalence,  $X$  can always be regarded as a delta-discrete  $G$ -spectrum, and thus,  $X$  can be thought of as having two types of homotopy fixed points,  $X^{hH}$  and  $(X_\delta)^{h_\delta H}$ , and, though its discrete homotopy fixed points  $X^{hH}$  are not known to always be well-behaved with respect to iteration, there is a reasonable alternative, the delta-discrete homotopy fixed points  $(X_\delta)^{h_\delta H}$ , which, thanks to (1.6), are always well-behaved.

As the reader might have noticed, given the work of [5] (as discussed earlier) and that of this paper, when  $G$  has finite vcd and  $X$  is a hyperfibrant discrete  $G$ -spectrum, there are two different ways to define an iterated homotopy fixed point spectrum: as  $(X^{hH})^{hK/H}$  and as  $(X^{hH})^{h_\delta K/H}$ . Though we are not able to show that these two objects are always equivalent, in Theorem 7.2, we show that if the canonical map  $(X_{fK})^H \rightarrow ((X_{fK})_{fH})^H$  is a weak equivalence, then there is a weak equivalence  $(X^{hH})^{hK/H} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}$ .

In the last section of this paper, Section 8, we show in two different, but interrelated ways that, for arbitrary  $G$ , the delta-discrete homotopy fixed point spectrum is always equivalent to a discrete homotopy fixed point spectrum. In each case, the equivalence is induced by a map between a discrete  $G$ -spectrum and a delta-discrete  $G$ -spectrum that need *not* be a weak equivalence. In Remark 8.7, we note several consequences of this observation for the categories  $c(\operatorname{Spt}_G)$  and  $c(\operatorname{Spt})$ , when each is equipped with the injective model structure.

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## 2. DELTA-DISCRETE $G$ -SPECTRA AND ITERATED HOMOTOPY FIXED POINTS

Let  $G$  be an arbitrary profinite group and  $X$  a discrete  $G$ -spectrum. In this section, we review (from [4]) the construction of a natural  $G$ -equivariant map  $\Psi : X \rightarrow X_\delta$  that is a weak equivalence in  $\operatorname{Spt}$ , giving a natural way of associating a delta-discrete  $G$ -spectrum to each object of  $\operatorname{Spt}_G$ . Also, we show that when  $G$  has finite vcd, then by using the framework of delta-discrete  $K/H$ -spectra, there is a sense in which homotopy fixed points of discrete  $G$ -spectra satisfy transitivity.

Now let  $G$  be any profinite group. There is a  $G$ -equivariant monomorphism  $i : X \rightarrow \operatorname{Map}_c(G, X)$  that is defined, on the level of sets, by  $i(x)(g) = g \cdot x$ , where

$x \in X_{k,l}$  and  $g \in G$ . Notice that  $i$  induces a  $G$ -equivariant map

$$\tilde{i}_X: \operatorname{holim}_{\Delta} \operatorname{cc}^{\bullet}(X) \rightarrow \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet}, X).$$

There is a natural  $G$ -equivariant map

$$\psi: X \cong \operatorname{colim}_{N \triangleleft_o G} X^N \rightarrow \operatorname{colim}_{N \triangleleft_o G} (X^N)_f = \widehat{X}$$

to the discrete  $G$ -spectrum  $\widehat{X}$ . We would like to know that  $\psi$  is a weak equivalence in  $\operatorname{Spt}_G$ ; however, the validity of this is not obvious, since  $\{X^N\}_{N \triangleleft_o G}$  is not known to be a diagram of fibrant spectra, and hence, we cannot use the fact that filtered colimits preserve weak equivalences between fibrant spectra. Nevertheless, the following lemma shows that it is still the case that  $\psi$  is a weak equivalence in  $\operatorname{Spt}_G$  (the author stated this without proof in [3, Remark 7.5] and [4, Definition 3.4]).

**Lemma 2.1.** If  $X$  is a discrete  $G$ -spectrum, then  $\psi: X \rightarrow \widehat{X}$  is a weak equivalence in  $\operatorname{Spt}_G$ .

*Proof.* Since  $X$  is a discrete  $G$ -spectrum,

$$\operatorname{Hom}_G(-, X): (G\text{-}\mathbf{Sets}_{df})^{\operatorname{op}} \rightarrow \operatorname{Spt}, \quad C \mapsto \operatorname{Hom}_G(C, X)$$

is a presheaf of spectra on the site  $G\text{-}\mathbf{Sets}_{df}$  of finite discrete  $G$ -sets (for more detail, we refer the reader to [3, Section 3] and [22, Sections 2.3, 6.2]). Here, the  $l$ -simplices of the  $k$ th simplicial set  $\operatorname{Hom}_G(C, X)_k$  are given by  $\operatorname{Hom}_G(C, X_{k,l})$ . Also, the composition

$$(-)_f \circ \operatorname{Hom}_G(-, X): (G\text{-}\mathbf{Sets}_{df})^{\operatorname{op}} \rightarrow \operatorname{Spt}, \quad C \mapsto (\operatorname{Hom}_G(C, X))_f$$

is a presheaf of spectra, and there is a map of presheaves

$$\widehat{\psi}: \operatorname{Hom}_G(-, X) \rightarrow (-)_f \circ \operatorname{Hom}_G(-, X)$$

that comes from the natural transformation  $\operatorname{id}_{\operatorname{Spt}} \rightarrow (-)_f$ .

Since  $\widehat{\psi}(C): \operatorname{Hom}_G(C, X) \rightarrow (\operatorname{Hom}_G(C, X))_f$  is a weak equivalence for every  $C \in G\text{-}\mathbf{Sets}_{df}$ , the map  $\pi_t(\widehat{\psi})$  of presheaves is an isomorphism for every integer  $t$ . Therefore, for every integer  $t$ ,  $\widetilde{\pi}_t(\widehat{\psi})$ , the map of sheaves associated to  $\pi_t(\widehat{\psi})$ , is an isomorphism, so that the map  $\widehat{\psi}$  is a local stable equivalence, and hence, a stalkwise weak equivalence. Thus, the map  $\operatorname{colim}_{N \triangleleft_o G} \widehat{\psi}(G/N)$  is a weak equivalence of spectra, and therefore, the map

$$\operatorname{colim}_{N \triangleleft_o G} X^N \cong \operatorname{colim}_{N \triangleleft_o G} \operatorname{Hom}_G(G/N, X) \xrightarrow{\cong} \operatorname{colim}_{N \triangleleft_o G} (\operatorname{Hom}_G(G/N, X))_f \cong \operatorname{colim}_{N \triangleleft_o G} (X^N)_f$$

is a weak equivalence, giving the desired conclusion.  $\square$

**Definition 2.2** ([4, pg. 145]). Given any profinite group  $G$  and any  $X \in \operatorname{Spt}_G$ , the composition

$$X \xrightarrow{\psi} \widehat{X} \xrightarrow{\cong} \lim_{\Delta} \operatorname{cc}^{\bullet}(\widehat{X}) \xrightarrow{\xi} \operatorname{holim}_{\Delta} \operatorname{cc}^{\bullet}(\widehat{X}) \xrightarrow{\tilde{i}_{\widehat{X}}} \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet}, \widehat{X}) = X_{\delta}$$

of natural maps, where the map  $\xi$  is the usual one (for example, see [19, Example 18.3.8, (2)]), defines the natural  $G$ -equivariant map

$$\Psi: X \rightarrow X_{\delta}$$

of spectra.



The next result was obtained in [4, pg. 145]; however, we give the proof below for completeness and because it is a key result.

**Lemma 2.3.** If  $G$  is any profinite group and  $X \in \text{Spt}_G$ , then the natural map  $\Psi: X \xrightarrow{\sim} X_\delta$  is a weak equivalence of spectra.

*Proof.* Following [4, pg. 145], there is a homotopy spectral sequence

$$E_2^{s,t} = \pi^s(\pi_t(\text{Map}_c(G^\bullet, \widehat{X}))) \Rightarrow \pi_{t-s}(X_\delta).$$

Let  $\pi_t(\text{Map}_c(G^\bullet, \widehat{X}))$  be the canonical cochain complex associated to the cosimplicial abelian group  $\pi_t(\text{Map}_c(G^\bullet, \widehat{X}))$ . As in [3, proof of Theorem 7.4], there is an exact sequence

$$0 \rightarrow \pi_t(\widehat{X}) \rightarrow \text{Map}_c(G^*, \pi_t(\widehat{X})) \cong \pi_t(\text{Map}_c(G^*, \widehat{X})),$$

so that  $E_2^{0,t} \cong \pi_t(\widehat{X}) \cong \pi_t(X)$ , where the last isomorphism is by Lemma 2.1, and  $E_2^{s,t} = 0$ , when  $s > 0$ . Thus, the above spectral sequence collapses, giving the desired result.  $\square$

Now we consider the iterated homotopy fixed points of discrete  $G$ -spectra by using the setting of delta-discrete  $K/H$ -spectra. Here, as in the Introduction,  $H$  and  $K$  are closed subgroups of  $G$ , with  $H \triangleleft K$ . We begin with a definition and some preliminary observations.

**Definition 2.4.** If  $Y^\bullet$  is a cosimplicial discrete  $G$ -spectrum, such that  $Y^n$  is fibrant in  $\text{Spt}_G$ , for each  $[n] \in \Delta$ , then we call  $Y^\bullet$  a *cosimplicial fibrant discrete  $G$ -spectrum*.

**Lemma 2.5** ([4, proof of Theorem 3.5]). If  $G$  is any profinite group and  $X \in \text{Spt}_G$ , then the cosimplicial discrete  $G$ -spectrum  $\text{Map}_c(G^\bullet, \widehat{X})$  is a cosimplicial fibrant discrete  $L$ -spectrum, for every closed subgroup  $L$  of  $G$ .

For the rest of this section, we assume that  $G$  has finite vcd and, as usual,  $X$  is a discrete  $G$ -spectrum. As in Lemma 2.5, let  $L$  be any closed subgroup of  $G$ . Then, by [3, Remark 7.13] and [4, Definition 5.1, Theorem 5.2] (the latter citation sets the former on a stronger footing), there is an identification

$$(2.6) \quad X^{hL} = \text{holim}_{\Delta} \text{Map}_c(G^\bullet, \widehat{X})^L.$$

(Notice that, under this identification,  $X^{hL} \cong (X_\delta)^L$ .)

**Lemma 2.7.** For each  $L$ , there is a weak equivalence  $X^{hL} \xrightarrow{\sim} (X_\delta)^{h_s L}$ .

*Proof.* By Lemma 2.5, the fibrant replacement map

$$\text{Map}_c(G^\bullet, \widehat{X})^n \xrightarrow{\sim} (\text{Map}_c(G^\bullet, \widehat{X})^n)_{fL}$$

is a weak equivalence between fibrant objects in  $\text{Spt}_L$ , for each  $[n] \in \Delta$ , so that

$$(\text{Map}_c(G^\bullet, \widehat{X})^n)^L \xrightarrow{\sim} ((\text{Map}_c(G^\bullet, \widehat{X})^n)_{fL})^L = (\text{Map}_c(G^\bullet, \widehat{X})^n)^{hL}$$

is a weak equivalence between fibrant objects in  $\text{Spt}$ . Thus, there is a weak equivalence

$$X^{hL} = \text{holim}_{\Delta} \text{Map}_c(G^\bullet, \widehat{X})^L \xrightarrow{\sim} (\text{holim}_{\Delta} \text{Map}_c(G^\bullet, \widehat{X}))^{h_s L} = (X_\delta)^{h_s L}.$$

$\square$

It will be useful to recall (see [5, Lemma 4.6]) that the functor

$$(-)^H : \text{Spt}_K \rightarrow \text{Spt}_{K/H}, \quad Y \mapsto Y^H$$

preserves fibrant objects.

Now we are ready for the task at hand. Since  $\text{Map}_c(G^\bullet, \widehat{X})^H$  is a cosimplicial discrete  $K/H$ -spectrum that is fibrant in  $\text{Spt}$  in each codegree, we can immediately conclude the following.

**Lemma 2.8.** The spectrum  $X^{hH} = \text{holim}_{\Delta} \text{Map}_c(G^\bullet, \widehat{X})^H$  is a delta-discrete  $K/H$ -spectrum.

Lemma 2.8 implies that

$$(X^{hH})^{h_\delta K/H} = \text{holim}_{[n] \in \Delta} ((\text{Map}_c(G^\bullet, \widehat{X})^n)^H)^{hK/H}.$$

**Theorem 2.9.** There is a weak equivalence  $X^{hK} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}$ .

*Proof.* Since  $\text{Map}_c(G^\bullet, \widehat{X})$  is a cosimplicial fibrant discrete  $K$ -spectrum, the diagram  $\text{Map}_c(G^\bullet, \widehat{X})^H$  is a cosimplicial fibrant discrete  $K/H$ -spectrum. Hence, each fibrant replacement map

$$(\text{Map}_c(G^\bullet, \widehat{X})^n)^H \xrightarrow{\simeq} ((\text{Map}_c(G^\bullet, \widehat{X})^n)^H)_{fK/H}$$

is a weak equivalence between fibrant objects in  $\text{Spt}_{K/H}$ , so that the induced map

$$(2.10) \quad \text{holim}_{[n] \in \Delta} ((\text{Map}_c(G^\bullet, \widehat{X})^n)^H)^{K/H} \xrightarrow{\simeq} \text{holim}_{[n] \in \Delta} (((\text{Map}_c(G^\bullet, \widehat{X})^n)^H)_{fK/H})^{K/H}$$

is a weak equivalence. The weak equivalence in (2.10) is exactly the weak equivalence

$$X^{hK} = \text{holim}_{[n] \in \Delta} (\text{Map}_c(G^\bullet, \widehat{X})^n)^K \xrightarrow{\simeq} \text{holim}_{[n] \in \Delta} ((\text{Map}_c(G^\bullet, \widehat{X})^n)^H)^{hK/H} = (X^{hH})^{h_\delta K/H}.$$

□

Lemma 2.8 and Theorem 2.9 show that, when  $G$  has finite vcd, the iterated homotopy fixed point spectrum  $(X^{hH})^{h_\delta K/H}$  is always defined and it is just  $X^{hK}$ .

### 3. THE HOMOTOPY FIXED POINTS OF DELTA-DISCRETE $G$ -SPECTRA AS A TOTAL RIGHT DERIVED FUNCTOR

Let  $G$  be any profinite group. In this section, we show that  $(-)^{h_\delta G}$ , the operation of taking the homotopy fixed points of a delta-discrete  $G$ -spectrum, can be viewed as the total right derived functor of

$$\lim_{\Delta} (-)^G : c(\text{Spt}_G) \rightarrow \text{Spt}, \quad X^\bullet \mapsto \lim_{[n] \in \Delta} (X^n)^G,$$

where  $c(\text{Spt}_G)$  has the injective model category structure (defined below). We will obtain this result as a special case of a more general result. Thus, we let  $\mathcal{C}$  denote any small category and we use “ $Z^\bullet$ ” and “ $X^\bullet$ ” to denote  $\mathcal{C}$ -shaped diagrams in  $\text{Spt}$  and  $\text{Spt}_G$ , respectively, since we are especially interested in the case when  $\mathcal{C} = \Delta$ .

Recall that  $\text{Spt}$  is a combinatorial model category (for the definition of this notion, we refer the reader to the helpful expositions in [9, Section 2] and [24, Section A.2.6 (and Definition A.1.1.2)]); this well-known fact is stated explicitly in [29, pg. 459]. Therefore, [24, Proposition A.2.8.2] implies that  $\text{Spt}^{\mathcal{C}}$ , the category of functors  $\mathcal{C} \rightarrow \text{Spt}$ , has an injective model category structure: more precisely,  $\text{Spt}^{\mathcal{C}}$

has a model structure in which a map  $f^\bullet: Z^\bullet \rightarrow W^\bullet$  in  $\text{Spt}^{\mathcal{C}}$  is a weak equivalence (cofibration) if and only if, for each  $C \in \mathcal{C}$ , the map  $f^C: Z^C \rightarrow W^C$  is a weak equivalence (cofibration) in  $\text{Spt}$ .

Similar comments apply to  $\text{Spt}_G$ . For example, the proof of [1, Theorem 2.2.1] applies [20, Definition 3.3] to obtain the model structure on  $\text{Spt}_G$ , and thus,  $\text{Spt}_G$  is a cellular model category. Hence,  $\text{Spt}_G$  is cofibrantly generated. Also, the category  $\text{Spt}_G$  is equivalent to the category of sheaves of spectra on the site  $G - \mathbf{Sets}_{df}$  (see [3, Section 3]), and thus,  $\text{Spt}_G$  is a locally presentable category, since standard arguments show that such a category of sheaves is locally presentable. (For example, see the general comment about such categories in [32, pg. 2]. The basic ideas are contained in the proof of the fact that a Grothendieck topos is locally presentable (see, for example, [2, Proposition 3.4.16]), and hence, the category of sheaves of sets on the aforementioned site is locally presentable.)

From the above considerations, we conclude that  $\text{Spt}_G$  is a combinatorial model category. Therefore, as before, [24, Proposition A.2.8.2] implies that the category  $(\text{Spt}_G)^{\mathcal{C}}$  of  $\mathcal{C}$ -shaped diagrams in  $\text{Spt}_G$  has an injective model structure in which a map  $h^\bullet$  is a weak equivalence (cofibration) if and only if each map  $h^C$  is a weak equivalence (cofibration) in  $\text{Spt}_G$ .

It will be useful to note that, by [24, Remark A.2.8.5], if  $f^\bullet$  is a fibration in  $\text{Spt}^{\mathcal{C}}$ , then  $f^C$  is a fibration in  $\text{Spt}$ , for every  $C \in \mathcal{C}$ . Similarly, if  $h^\bullet$  is a fibration in  $(\text{Spt}_G)^{\mathcal{C}}$ , then each map  $h^C$  is a fibration in  $\text{Spt}_G$ .

The functor

$$\lim_{\mathcal{C}}(-)^G: (\text{Spt}_G)^{\mathcal{C}} \rightarrow \text{Spt}, \quad X^\bullet \mapsto \lim_{C \in \mathcal{C}} (X^C)^G$$

is right adjoint to the functor  $\underline{t}: \text{Spt} \rightarrow (\text{Spt}_G)^{\mathcal{C}}$  that sends an arbitrary spectrum  $Z$  to the constant  $\mathcal{C}$ -shaped diagram  $\underline{t}(Z)$  on the discrete  $G$ -spectrum  $t(Z)$ , where (as in [3, Corollary 3.9])

$$t: \text{Spt} \rightarrow \text{Spt}_G, \quad Z \mapsto t(Z) = Z$$

is the functor that equips  $Z$  with the trivial  $G$ -action. If  $f$  is a weak equivalence (cofibration) in  $\text{Spt}$ , then  $t(f)$  is a weak equivalence (cofibration) in  $\text{Spt}_G$ , and hence,  $\underline{t}(f)$  is a weak equivalence (cofibration) in  $(\text{Spt}_G)^{\mathcal{C}}$ . This observation immediately gives the following result.

**Lemma 3.1.** The functors  $(\underline{t}, \lim_{\mathcal{C}}(-)^G)$  are a Quillen pair for  $(\text{Spt}, (\text{Spt}_G)^{\mathcal{C}})$ .

We let

$$(-)_{fib}: (\text{Spt}_G)^{\mathcal{C}} \rightarrow (\text{Spt}_G)^{\mathcal{C}}, \quad X^\bullet \mapsto (X^\bullet)_{fib}$$

denote a fibrant replacement functor, such that there is a morphism  $X^\bullet \rightarrow (X^\bullet)_{fib}$  in  $(\text{Spt}_G)^{\mathcal{C}}$  that is a natural trivial cofibration, with  $(X^\bullet)_{fib}$  fibrant, in  $(\text{Spt}_G)^{\mathcal{C}}$ . Then Lemma 3.1 implies the existence of the total right derived functor

$$\mathbf{R}(\lim_{\mathcal{C}}(-)^G): \text{Ho}((\text{Spt}_G)^{\mathcal{C}}) \rightarrow \text{Ho}(\text{Spt}), \quad X^\bullet \mapsto \lim_{\mathcal{C}}((X^\bullet)_{fib})^G.$$

Now we prove the key result that will allow us to relate Definition 1.5 to the total right derived functor  $\mathbf{R}(\lim_{\Delta}(-)^G)$ .

**Theorem 3.2.** Given  $X^\bullet$  in  $(\text{Spt}_G)^{\mathcal{C}}$ , the canonical map

$$(\mathbf{R}(\lim_{\mathcal{C}}(-)^G))(X^\bullet) = \lim_{\mathcal{C}}((X^\bullet)_{fib})^G \xrightarrow{\simeq} \text{holim}_{\mathcal{C}}((X^\bullet)_{fib})^G$$

is a weak equivalence of spectra.

*Proof.* Let  $t^{\mathcal{C}}: \text{Spt}^{\mathcal{C}} \rightarrow (\text{Spt}_G)^{\mathcal{C}}$  be the functor that sends  $Z^{\bullet}$  to  $Z^{\bullet}$ , where each  $Z^C$  is regarded as having the trivial  $G$ -action. Then  $t^{\mathcal{C}}$  preserves weak equivalences and cofibrations, so that the right adjoint of  $t^{\mathcal{C}}$ , the fixed points functor  $(-)^G: (\text{Spt}_G)^{\mathcal{C}} \rightarrow \text{Spt}^{\mathcal{C}}$ , preserves fibrant objects. Thus, the diagram  $((X^{\bullet})_{\mathcal{Fib}})^G$  is fibrant in  $\text{Spt}^{\mathcal{C}}$ .

Let  $Z^{\bullet}$  be a fibrant object in  $\text{Spt}^{\mathcal{C}}$ . Then, to complete the proof, it suffices to show that the following canonical map is a weak equivalence:

$$\lambda: \lim_{\mathcal{C}} Z^{\bullet} \rightarrow \text{holim}_{\mathcal{C}} Z^{\bullet}.$$

(Though this assertion seems to be well-known, for the sake of completeness, we justify it below. Also, see Remark 8.9.) Since the functor  $\text{Spt} \rightarrow \text{Spt}^{\mathcal{C}}$  that sends a spectrum  $W$  to the constant  $\mathcal{C}$ -shaped diagram on  $W$  preserves weak equivalences and cofibrations, its right adjoint, the functor  $\lim_{\mathcal{C}}(-): \text{Spt}^{\mathcal{C}} \rightarrow \text{Spt}$ , preserves fibrant objects. Thus,  $\lim_{\mathcal{C}} Z^{\bullet}$ , the source of  $\lambda$ , is a fibrant spectrum.

Since  $Z^{\bullet}$  is fibrant in  $\text{Spt}^{\mathcal{C}}$ ,  $Z^C$  is fibrant in  $\text{Spt}$ , for each  $C \in \mathcal{C}$ , and hence,  $\text{holim}_{\mathcal{C}} Z^{\bullet}$  is a fibrant spectrum. Therefore, since the source and target of  $\lambda$  are fibrant spectra, to verify that  $\lambda$  is a weak equivalence, we only have to show that the map

$$\lambda_k: \lim_{\mathcal{C}} (Z^{\bullet})_k = (\lim_{\mathcal{C}} Z^{\bullet})_k \rightarrow (\text{holim}_{\mathcal{C}} Z^{\bullet})_k = \text{holim}_{\mathcal{C}} (Z^{\bullet})_k$$

is a weak equivalence of simplicial sets, for each  $k \geq 0$ , where the limit and homotopy limit of  $(Z^{\bullet})_k$  in the source and target, respectively, of  $\lambda_k$  are formed in  $\mathcal{S}$ , the category of simplicial sets. To do this, we equip  $\mathcal{S}^{\mathcal{C}}$ , the category of  $\mathcal{C}$ -shaped diagrams of simplicial sets, with an injective model structure, so that a morphism  $f^{\bullet}: K^{\bullet} \rightarrow L^{\bullet}$  in  $\mathcal{S}^{\mathcal{C}}$  is a weak equivalence (cofibration) if and only if  $f^C: K^C \rightarrow L^C$  is a weak equivalence (cofibration) of simplicial sets, for each  $C \in \mathcal{C}$  (the injective model structure on  $\mathcal{S}^{\mathcal{C}}$  exists, for example, by [16, pg. 403, Proposition 2.4]).

Notice that the category  $\text{Spt}^{\mathcal{C}}$  is equal to the category  $\text{Spt}^{(\mathcal{C}^{\text{op}})^{\text{op}}}$  of presheaves of spectra on  $\mathcal{C}^{\text{op}}$ , where  $\mathcal{C}^{\text{op}}$  is regarded as a site by equipping it with the chaotic topology. Then, by [22, Remark 2.36], the injective model structure on  $\text{Spt}^{\mathcal{C}}$  is exactly the local injective model structure (of [21]) on the category  $\text{Spt}^{(\mathcal{C}^{\text{op}})^{\text{op}}}$  of presheaves of spectra on the site  $\mathcal{C}^{\text{op}}$ . Thus, by [22, Remark 2.35],  $(Z^{\bullet})_k$  is a globally fibrant simplicial presheaf of sets on the site  $\mathcal{C}^{\text{op}}$ , which means exactly that  $(Z^{\bullet})_k$  is fibrant in  $\mathcal{S}^{\mathcal{C}}$ . This conclusion implies that  $\lambda_k$  is a weak equivalence, by the proof of [16, pg. 407, Lemma 2.11], giving the desired result.  $\square$

Given  $X^{\bullet} \in (\text{Spt}_G)^{\mathcal{C}}$ , we let  $((X^{\bullet})_{\mathcal{Fib}})^{\mathcal{C}}$  denote the value of  $(X^{\bullet})_{\mathcal{Fib}}: \mathcal{C} \rightarrow \text{Spt}_G$  on the object  $C \in \mathcal{C}$ .

**Lemma 3.3.** If  $X^{\bullet}$  is an object in  $(\text{Spt}_G)^{\mathcal{C}}$ , then there is a weak equivalence

$$\text{holim}_{C \in \mathcal{C}} (X^C)^{hG} \xrightarrow{\sim} \text{holim}_{C \in \mathcal{C}} (((X^{\bullet})_{\mathcal{Fib}})^C)^G.$$

*Proof.* Let  $(X^{\bullet})_{fG}$  be the object in  $(\text{Spt}_G)^{\mathcal{C}}$  that is equal to the composition of functors

$$(-)_{fG} \circ (X^{\bullet}): \mathcal{C} \rightarrow \text{Spt}_G, \quad C \mapsto (X^C)_{fG}.$$

Since  $X^{\bullet} \rightarrow (X^{\bullet})_{fG}$  is a trivial cofibration in  $(\text{Spt}_G)^{\mathcal{C}}$ , the fibrant object  $(X^{\bullet})_{\mathcal{Fib}}$  induces a weak equivalence

$$\ell^{\bullet}: (X^{\bullet})_{fG} \xrightarrow{\sim} (X^{\bullet})_{\mathcal{Fib}}$$

in  $(\text{Spt}_G)^{\mathcal{C}}$ . Therefore, since  $(X^C)_{fG}$  and  $((X^\bullet)_{fib})^C$  are fibrant discrete  $G$ -spectra, for each  $C \in \mathcal{C}$ , there is a weak equivalence

$$\text{holim}_{C \in \mathcal{C}} (\ell^C)^G : \text{holim}_{C \in \mathcal{C}} (X^C)^{hG} = \text{holim}_{C \in \mathcal{C}} ((X^C)_{fG})^G \xrightarrow{\cong} \text{holim}_{C \in \mathcal{C}} (((X^\bullet)_{fib})^C)^G.$$

□

By letting  $\mathcal{C} = \Delta$ , Theorem 3.2 and Lemma 3.3 immediately yield the next result, which allows us to conclude that *homotopy* fixed points for delta-discrete  $G$ -spectra,  $(-)^{h_\delta G}$ , can indeed be regarded as the total right derived functor of fixed points, in the appropriate sense.

**Theorem 3.4.** If  $\text{holim}_\Delta X^\bullet$  is a delta-discrete  $G$ -spectrum, then there is a zigzag

$$(\text{holim}_\Delta X^\bullet)^{h_\delta G} \xrightarrow{\cong} \text{holim}_\Delta ((X^\bullet)_{fib})^G \xleftarrow{\cong} (\mathbf{R}(\lim_{\Delta} (-)^G))(X^\bullet)$$

of weak equivalences in  $\text{Spt}$ .

#### 4. SEVERAL PROPERTIES OF THE HOMOTOPY FIXED POINTS OF DELTA-DISCRETE $G$ -SPECTRA

Suppose that  $P$  is a finite group and let  $Z$  be a  $P$ -spectrum. Recall (for example, from [3, Section 5]) that if  $Z'$  is a  $P$ -spectrum and a fibrant object in  $\text{Spt}$ , with a map  $Z \xrightarrow{\cong} Z'$  that is  $P$ -equivariant and a weak equivalence in  $\text{Spt}$ , then  $Z^{h'P}$ , the usual homotopy fixed point spectrum  $\text{Map}_P(EP_+, Z')$  in the case when  $P$  is a finite discrete group, can also be defined as

$$(4.1) \quad Z^{h'P} = \text{holim}_P Z'.$$

Then the following result shows that the homotopy fixed points  $(-)^{h_\delta G}$  of Definition 1.5 agree with those of (4.1), when the profinite group  $G$  is finite and discrete.

**Theorem 4.2.** Let  $G$  be a finite discrete group and let  $\text{holim}_\Delta X^\bullet$  be a delta-discrete  $G$ -spectrum (that is,  $X^\bullet$  is a cosimplicial  $G$ -spectrum, with each  $X^n$  a fibrant spectrum). Then there is a weak equivalence

$$(\text{holim}_\Delta X^\bullet)^{h_\delta G} \xrightarrow{\cong} (\text{holim}_\Delta X^\bullet)^{h'G}.$$

*Proof.* Given  $[n] \in \Delta$ , by [22, Proposition 6.39], the canonical map

$$(X^n)^{hG} = ((X^n)_{fG})^G \cong \lim_G (X^n)_{fG} \xrightarrow{\cong} \text{holim}_G (X^n)_{fG}$$

is a weak equivalence. Also, notice that the target (since  $(X^n)_{fG}$  is a fibrant spectrum, by [3, Lemma 3.10]) and the source of this weak equivalence are fibrant spectra. Thus, there is a weak equivalence

$$(\text{holim}_\Delta X^\bullet)^{h_\delta G} = \text{holim}_{[n] \in \Delta} (X^n)^{hG} \xrightarrow{\cong} \text{holim}_{[n] \in \Delta} \text{holim}_G (X^n)_{fG} \cong \text{holim}_G \text{holim}_{[n] \in \Delta} (X^n)_{fG}.$$

The proof is finished by noting that

$$\text{holim}_G \text{holim}_{[n] \in \Delta} (X^n)_{fG} = (\text{holim}_\Delta X^\bullet)^{h'G};$$

this equality, which is an application of (4.1), is due to the fact that the map  $\text{holim}_\Delta X^\bullet \rightarrow \text{holim}_{[n] \in \Delta} (X^n)_{fG}$  is  $G$ -equivariant and a weak equivalence (since each map  $X^n \rightarrow (X^n)_{fG}$  is a weak equivalence between fibrant objects in  $\text{Spt}$ ), with target a fibrant spectrum. □

Now let  $G$  be any profinite group and let  $X$  be a discrete  $G$ -spectrum. We will show that there is a  $G$ -equivariant map  $X \rightarrow \mathrm{cc}_G(X)$  that is a weak equivalence of spectra, along with a weak equivalence  $X^{hG} \rightarrow (\mathrm{cc}_G(X))^{h_\delta G}$ . Since these weak equivalences exist for any  $G$  and all  $X \in \mathrm{Spt}_G$ , we can think of the world of delta-discrete  $G$ -spectra as being a generalization of the category  $\mathrm{Spt}_G$ .

Given any spectrum  $Z$ , it is not hard to see that there is an isomorphism

$$\mathrm{Tot}(\mathrm{cc}^\bullet(Z)) \cong Z;$$

this was noted, for example, in the setting of simplicial sets, in [12, Section 1] and is verified for an arbitrary simplicial model category in [18, Remark B.16]. Since the Reedy category  $\Delta$  has fibrant constants (see [19, Corollary 15.10.5]), the canonical map  $\mathrm{Tot}(\mathrm{cc}^\bullet(Z)) \rightarrow \mathrm{holim}_\Delta \mathrm{cc}^\bullet(Z)$  is a weak equivalence, whenever  $Z$  is a fibrant spectrum, by [19, Theorem 18.7.4, (2)]. Thus, if  $Z$  is a fibrant spectrum, there is a weak equivalence

$$\phi_Z: Z \cong \mathrm{Tot}(\mathrm{cc}^\bullet(Z)) \xrightarrow{\sim} \mathrm{holim}_\Delta \mathrm{cc}^\bullet(Z)$$

in  $\mathrm{Spt}$ . In particular, since the discrete  $G$ -spectrum  $\widehat{X}$  is a fibrant spectrum, the map  $\phi_{\widehat{X}}$  is a weak equivalence. Therefore, since the map  $\psi: X \rightarrow \widehat{X}$  is a weak equivalence in  $\mathrm{Spt}_G$  (by Lemma 2.1), the  $G$ -equivariant map

$$\phi_{\widehat{X}} \circ \psi: X \xrightarrow{\sim} \widehat{X} \xrightarrow{\sim} \mathrm{holim}_\Delta \mathrm{cc}^\bullet(\widehat{X}) = \mathrm{cc}_G(X)$$

and the map

$$\phi_{(\widehat{X})^{hG}} \circ (\psi)^{hG}: X^{hG} \xrightarrow{\sim} (\widehat{X})^{hG} \xrightarrow{\sim} \mathrm{holim}_\Delta \mathrm{cc}^\bullet((\widehat{X})^{hG}) = (\mathrm{cc}_G(X))^{h_\delta G}$$

are weak equivalences (the map  $\phi_{(\widehat{X})^{hG}}$  is a weak equivalence because  $(\widehat{X})^{hG}$  is a fibrant spectrum).

## 5. ITERATED HOMOTOPY FIXED POINTS FOR DELTA-DISCRETE $G$ -SPECTRA

Throughout this section (except in Convention 5.1), we assume that the profinite group  $G$  has finite vcd. We will show that  $G$ -homotopy fixed points for delta-discrete  $G$ -spectra have the transitivity property. To do this, we make use of the convention stated below.

**Convention 5.1.** Let  $P$  be a discrete group and let  $\mathcal{X}$  be a space. By [11, Remark 10.3], a *proxy action* of  $P$  on  $\mathcal{X}$  is a space  $\mathcal{Y}$  that is homotopy equivalent to  $\mathcal{X}$  and has an action of  $P$ . Then in [11, Remark 10.3], Dwyer and Wilkerson establish the convention that  $\mathcal{X}^{hP}$  is *equal* to  $\mathcal{Y}^{hP}$ , and a proxy action is sometimes referred to as an action. This convention is an important one: for example, in [11, Section 10], this convention plays a role in Lemmas 10.4 and 10.6 and in the proof that  $P$ -homotopy fixed points have the transitivity property (their Lemma 10.5). Thus, in this section, we will make use of the related convention described below.

Let  $G$  be any profinite group and let  $X^{\bullet,\bullet}$  be a bicosimplicial discrete  $G$ -spectrum (that is,  $X^{\bullet,\bullet}$  is a cosimplicial object in  $c(\mathrm{Spt}_G)$ ), such that, for all  $m, n \geq 0$ ,  $X^{m,n}$  is a fibrant spectrum. Let  $\{X^{n,n}\}_{[n] \in \Delta}$  be the cosimplicial discrete  $G$ -spectrum that is the diagonal of  $X^{\bullet,\bullet}$ ;  $\{X^{n,n}\}_{[n] \in \Delta}$  is defined to be the composition

$$\Delta \rightarrow \Delta \times \Delta \rightarrow \mathrm{Spt}_G, \quad [n] \mapsto ([n], [n]) \mapsto X^{n,n}.$$

Then there is a natural  $G$ -equivariant map

$$(5.2) \quad \operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet} \xrightarrow{\sim} \operatorname{holim}_{[n] \in \Delta} X^{n, n}$$

that is a weak equivalence (see, for example, [31, Lemma 5.33] and [19, Remark 19.1.6, Theorem 19.6.7, (2)]). Notice that the target of (5.2),  $\operatorname{holim}_{[n] \in \Delta} X^{n, n}$ , is a delta-discrete  $G$ -spectrum. Thus, we identify the source of (5.2), the  $G$ -spectrum  $\operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet}$ , with the delta-discrete  $G$ -spectrum  $\operatorname{holim}_{[n] \in \Delta} X^{n, n}$ , so that

$$(5.3) \quad (\operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet})^{h_\delta G} := (\operatorname{holim}_{[n] \in \Delta} X^{n, n})^{h_\delta G}.$$

Let  $\operatorname{holim}_{\Delta \times \Delta} (X^{\bullet, \bullet})^{hG}$  denote  $\operatorname{holim}_{([m], [n]) \in \Delta \times \Delta} (X^{m, n})^{hG}$  and notice that, by Theorem 3.2 and Lemma 3.3, there is a zigzag of weak equivalences

$$\operatorname{holim}_{\Delta \times \Delta} (X^{\bullet, \bullet})^{hG} \xrightarrow{\sim} \operatorname{holim}_{\Delta \times \Delta} ((X^{\bullet, \bullet})_{\text{fib}})^G \xleftarrow{\sim} (\mathbf{R}(\lim_{\Delta \times \Delta} (-)^G))(X^{\bullet, \bullet}).$$

Hence, it is natural to define the homotopy fixed points of the “ $(\Delta \times \Delta)$ -discrete  $G$ -spectrum”  $\operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet}$  as

$$(\operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet})^{hG} = \operatorname{holim}_{\Delta \times \Delta} (X^{\bullet, \bullet})^{hG}.$$

Since each  $(X^{m, n})^{hG}$  is a fibrant spectrum, then, as in (5.2), there is a weak equivalence

$$(\operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet})^{hG} \xrightarrow{\sim} \operatorname{holim}_{[n] \in \Delta} (X^{n, n})^{hG} = (\operatorname{holim}_{\Delta \times \Delta} X^{\bullet, \bullet})^{h_\delta G},$$

which further justifies the convention given in (5.3).  $\square$

As in the Introduction, let  $H$  and  $K$  be closed subgroups of  $G$ , with  $H$  normal in  $K$ . Recall from (2.6) that, given  $X \in \operatorname{Spt}_G$ , there is an identification

$$X^{hH} = \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^{\bullet}, \widehat{X})^H.$$

Since the map  $\psi: X \rightarrow \widehat{X}$  is natural, it is clear from [4] (to be specific, in [4], see pg. 145 and the proof of Theorem 5.2) that the above identification is natural in  $X$ .

Now let  $\operatorname{holim}_{\Delta} X^{\bullet}$  be any delta-discrete  $G$ -spectrum. Using the naturality of the above identification, we have

$$(\operatorname{holim}_{\Delta} X^{\bullet})^{h_\delta H} = \operatorname{holim}_{[n] \in \Delta} (X^n)^{hH} = \operatorname{holim}_{[n] \in \Delta} \operatorname{holim}_{[m] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^m)^H.$$

Because of the isomorphism

$$\operatorname{holim}_{[n] \in \Delta} \operatorname{holim}_{[m] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^m)^H \cong \operatorname{holim}_{\Delta \times \Delta} \operatorname{Map}_c(G^{\bullet}, \widehat{X^{\bullet}})^H$$

and because homotopy limits are ends, which are only unique up to isomorphism, we can set

$$(\operatorname{holim}_{\Delta} X^{\bullet})^{h_\delta H} = \operatorname{holim}_{\Delta \times \Delta} \operatorname{Map}_c(G^{\bullet}, \widehat{X^{\bullet}})^H.$$

By Lemma 2.5, for each  $m, n \geq 0$ ,  $\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^m$  is a fibrant discrete  $H$ -spectrum, so that  $(\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^m)^H$  is a fibrant spectrum. Also, since the diagram  $\operatorname{Map}_c(G^{\bullet}, \widehat{X^{\bullet}})$  is a bicosimplicial discrete  $K$ -spectrum,  $\operatorname{Map}_c(G^{\bullet}, \widehat{X^{\bullet}})^H$  is a

bicosimplicial discrete  $K/H$ -spectrum. Thus, the discussion above in Convention 5.1 implies that there is a  $K/H$ -equivariant map

$$(\operatorname{holim}_{\Delta} X^{\bullet})^{h_{\delta}H} = \operatorname{holim}_{\Delta \times \Delta} \operatorname{Map}_c(G^{\bullet}, \widehat{X^{\bullet}})^H \xrightarrow{\simeq} \operatorname{holim}_{[n] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^n)^H$$

that is a weak equivalence, and the target of this weak equivalence is a delta-discrete  $K/H$ -spectrum. Therefore, by Convention 5.1, we can identify  $(\operatorname{holim}_{\Delta} X^{\bullet})^{h_{\delta}H}$  with the delta-discrete  $K/H$ -spectrum  $\operatorname{holim}_{[n] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^n)^H$ , and hence, by (5.3) and as in the proof of Theorem 2.9, we have

$$\begin{aligned} ((\operatorname{holim}_{\Delta} X^{\bullet})^{h_{\delta}H})^{h_{\delta}K/H} &= (\operatorname{holim}_{[n] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^n)^H)^{h_{\delta}K/H} \\ &= \operatorname{holim}_{[n] \in \Delta} ((\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^n)^H)_{fK/H}^{K/H} \\ &\xleftarrow{\simeq} \operatorname{holim}_{[n] \in \Delta} ((\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^n)^H)^{K/H} \\ &= \operatorname{holim}_{[n] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^n)^K \\ &\xleftarrow{\simeq} \operatorname{holim}_{[n] \in \Delta} \operatorname{holim}_{[m] \in \Delta} (\operatorname{Map}_c(G^{\bullet}, \widehat{X^n})^m)^K \\ &= \operatorname{holim}_{[n] \in \Delta} (X^n)^{hK} \\ &= (\operatorname{holim}_{\Delta} X^{\bullet})^{h_{\delta}K}. \end{aligned}$$

We summarize our work above in the following theorem.

**Theorem 5.4.** If  $G$  has finite vcd and  $\operatorname{holim}_{\Delta} X^{\bullet}$  is a delta-discrete  $G$ -spectrum, then there is a weak equivalence

$$((\operatorname{holim}_{\Delta} X^{\bullet})^{h_{\delta}H})^{h_{\delta}K/H} \xleftarrow{\simeq} (\operatorname{holim}_{\Delta} X^{\bullet})^{h_{\delta}K}.$$

## 6. THE RELATIONSHIP BETWEEN $X^{hH}$ AND $(X_{\delta})^{h_{\delta}H}$ , IN GENERAL

Let  $G$  be an arbitrary profinite group and let  $X$  be any discrete  $G$ -spectrum. Also, as usual, let  $H$  and  $K$  be closed subgroups of  $G$ , with  $H$  normal in  $K$ . As mentioned near the beginning of the Introduction, it is not known, in general, that the “discrete homotopy fixed points”  $X^{hH}$  have a  $K/H$ -action. In this section, we consider this issue by using the framework of delta-discrete  $G$ -spectra.

Given the initial data above, there is a commutative diagram

$$\begin{array}{ccccc} X_{fH} & \xleftarrow[\simeq]{f_H} & X & \xrightarrow[\simeq]{\Psi} & X_{\delta} \\ & \searrow \Psi_H & \downarrow \operatorname{colim}_{U \triangleleft_o H} \Psi^U & \nearrow & \\ & & \operatorname{colim}_{U \triangleleft_o H} (X_{\delta})^U & & \end{array}$$

where  $\operatorname{colim}_{U \triangleleft_o H} \Psi^U$ , a morphism in  $\operatorname{Spt}_H$  whose label is a slight abuse of notation, is defined to be the composition

$$X \cong \operatorname{colim}_{U \triangleleft_o H} X^U \rightarrow \operatorname{colim}_{U \triangleleft_o H} (X_{\delta})^U,$$



and  $\Psi_H$ , a morphism between fibrant objects in  $\text{Spt}_H$ , exists because, in  $\text{Spt}_H$ ,  $f_H$  is a trivial cofibration and  $\text{colim}_{U \triangleleft_o H} (X_\delta)^U$  is fibrant (by [4, Theorem 3.5]).

**Definition 6.1.** The map  $\Psi_H$  induces the map

$$X^{hH} = (X_{fH})^H \xrightarrow{(\Psi_H)^H} (\text{colim}_{U \triangleleft_o H} (X_\delta)^U)^H \cong \text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^H$$

and (as in the proof of Lemma 2.7) there is a weak equivalence

$$\text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^H \xrightarrow{\simeq} \text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^{hH} = (X_\delta)^{h_\delta H}.$$

The composition of these two maps defines the map

$$\rho(X)_H : X^{hH} \rightarrow (X_\delta)^{h_\delta H}.$$

(The map  $\rho(X)_H$  is not the same as the map of Lemma 2.7 (when  $L = H$ ), since  $\rho(X)_H$  does not use the identification of (2.6).)

Notice that if the map  $\text{colim}_{U \triangleleft_o H} \Psi^U$  is a weak equivalence in  $\text{Spt}$ , then the map  $\Psi_H$  is a weak equivalence in  $\text{Spt}_H$ , and hence,  $(\Psi_H)^H$  is a weak equivalence. Thus, if  $\text{colim}_{U \triangleleft_o H} \Psi^U$  is a weak equivalence in  $\text{Spt}$ , then  $\rho(X)_H$  is a weak equivalence. This observation, together with [4, proof of Theorem 4.2] and [25, Proposition 3.3], immediately yields the following result.

**Theorem 6.2.** If  $G$  is any profinite group and  $X \in \text{Spt}_G$ , then the map

$$\rho(X)_H : X^{hH} \xrightarrow{\simeq} (X_\delta)^{h_\delta H}$$

is a weak equivalence, whenever any one of the following conditions holds:

- (i)  $H$  has finite vcd;
- (ii)  $G$  has finite vcd;
- (iii) there exists a fixed integer  $p$  such that  $H_c^s(U; \pi_t(X)) = 0$ , for all  $s > p$ , all  $t \in \mathbb{Z}$ , and all  $U \triangleleft_o H$ ;
- (iv) there exists a fixed integer  $q$  such that  $H_c^s(U; \pi_t(X)) = 0$ , for all  $t > q$ , all  $s \geq 0$ , and all  $U \triangleleft_o H$ ; or
- (v) there exists a fixed integer  $r$  such that  $\pi_t(X) = 0$ , for all  $t > r$ .

In the statement of Theorem 6.2, note that (ii) implies (i) and (v) implies (iv). Also, it is not known, in general, that  $\text{colim}_{U \triangleleft_o H} \Psi^U$  is a weak equivalence, so that we do not know, in general, that  $X^{hH}$  and  $(X_\delta)^{h_\delta H}$  are equivalent.

As noted in Definition 6.1,  $(X_\delta)^{h_\delta H}$  is equivalent to the delta-discrete  $K/H$ -spectrum  $\text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^H$ , and hence, it is natural to identify them and to set

$$((X_\delta)^{h_\delta H})^{h_\delta K/H} = (\text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^H)^{h_\delta K/H}.$$

**Theorem 6.3.** If  $G$  is any profinite group and  $X \in \text{Spt}_G$ , then

$$((X_\delta)^{h_\delta H})^{h_\delta K/H} \simeq (X_\delta)^{h_\delta K}.$$

*Proof.* As in the proof of Theorem 2.9, it is easy to see that there is a zigzag of weak equivalences

$$((X_\delta)^{h_\delta H})^{h_\delta K/H} \xleftarrow{\simeq} \text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^K \xrightarrow{\simeq} (X_\delta)^{h_\delta K}.$$

□

Given any profinite group  $G$ , Theorem 6.3 shows that, by using  $(-)_\delta$  and  $(-)^{h_\delta H}$ , delta-discrete homotopy fixed points for discrete  $G$ -spectra are transitive.

## 7. COMPARING TWO DIFFERENT MODELS FOR THE ITERATED HOMOTOPY FIXED POINT SPECTRUM

Let  $G$  be any profinite group and let  $X$  be a discrete  $G$ -spectrum. Recall from [5, Definition 4.1] that  $X$  is a hyperfibrant discrete  $G$ -spectrum if the map

$$\psi(X)_L^G: (X_{fG})^L \rightarrow ((X_{fG})_{fL})^L$$

is a weak equivalence for every closed subgroup  $L$  of  $G$ . (We use “ $\psi$ ” in the notation “ $\psi(X)_L^G$ ,” because we follow the notation of [5]; this use of “ $\psi$ ” is not related to the map  $\psi$  of Lemma 2.1.)

Now suppose that  $X$  is a hyperfibrant discrete  $G$ -spectrum and, as usual, let  $H$  and  $K$  be closed subgroups of  $G$ , with  $H$  normal in  $K$ . These hypotheses imply that

- (a) the map  $\psi(X)_H^G: (X_{fG})^H \xrightarrow{\simeq} ((X_{fG})_{fH})^H$  is a weak equivalence;
- (b) the source of the map  $\psi(X)_H^G$ , the spectrum  $(X_{fG})^H$ , is a discrete  $K/H$ -spectrum; and
- (c) since the composition  $X \rightarrow X_{fG} \rightarrow (X_{fG})_{fH}$  is a trivial cofibration and the target of the weak equivalence  $X \rightarrow X_{fH}$  is fibrant, in  $\text{Spt}_H$ , there is a weak equivalence  $v: (X_{fG})_{fH} \rightarrow X_{fH}$  between fibrant objects, and hence, there is a weak equivalence  $(X_{fG})^H \xrightarrow{\simeq} X^{hH}$  that is defined by the composition

$$v^H \circ \psi(X)_H^G: (X_{fG})^H \xrightarrow{\simeq} ((X_{fG})_{fH})^H \xrightarrow{\simeq} (X_{fH})^H = X^{hH}.$$

Thus, following [5, Definition 4.5], it is natural to define

$$(X^{hH})^{hK/H} := ((X_{fG})^H)^{hK/H}.$$

Let  $G$  have finite vcd, so that, by Lemma 2.8,  $X^{hH} = \text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^H$  is a delta-discrete  $K/H$ -spectrum. Thus,

$$(X^{hH})^{h_\delta K/H} = \text{holim}_{[n] \in \Delta} ((\text{Map}_c(G^\bullet, \widehat{X})^n)^H)^{hK/H},$$

and, by Theorem 2.9, there is a weak equivalence

$$(7.1) \quad \text{holim}_\Delta \text{Map}_c(G^\bullet, \widehat{X})^K \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

The above discussion shows that when  $G$  has finite vcd and  $X$  is a hyperfibrant discrete  $G$ -spectrum, there are two different models for the iterated homotopy fixed point spectrum,  $(X^{hH})^{hK/H}$  and  $(X^{hH})^{h_\delta K/H}$ , and we would like to know when they agree with each other. The following result gives a criterion for when  $(X^{hH})^{hK/H}$  and  $(X^{hH})^{h_\delta K/H}$  are equivalent.

**Theorem 7.2.** Let  $G$  have finite vcd and suppose that  $X$  is a hyperfibrant discrete  $G$ -spectrum. If the map  $\psi(X)_H^K: (X_{fK})^H \rightarrow ((X_{fK})_{fH})^H$  is a weak equivalence, then there is a weak equivalence

$$(7.3) \quad (X^{hH})^{hK/H} \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

**Remark 7.4.** Let  $G$  and  $X$  be as in Theorem 7.2. By definition, if  $X$  is a hyperfibrant discrete  $K$ -spectrum, then each map  $\psi(X)_H^K$  is a weak equivalence (here, as usual,  $H$  is any closed subgroup of  $G$  that is normal in  $K$ ), giving the weak equivalence of (7.3). We refer the reader to [5, Sections 3, 4] for further discussion about hyperfibrancy. The spectral sequence considerations of [5, pp. 2887–2888] are helpful for understanding when  $X$  is a hyperfibrant discrete  $K$ -spectrum.

*Proof of Theorem 7.2.* It follows from Theorem 6.2, (ii) and Definition 6.1 that  $(\Psi_K)^K$  is a weak equivalence, so that, by composing with the weak equivalence of (7.1), there is a weak equivalence

$$X^{hK} \xrightarrow{\simeq} \operatorname{holim}_{\Delta} \operatorname{Map}_c(G^\bullet, \widehat{X})^K \xrightarrow{\simeq} (X^{hH})^{h_\delta K/H}.$$

Therefore, to obtain (7.3), it suffices to show that there is a weak equivalence

$$(X^{hH})^{hK/H} = ((X_{fG})^H)^{hK/H} \rightarrow X^{hK}.$$

It is useful for the argument below to recall from the proof of [5, Lemma 4.6] that the functor  $(-)^H: \operatorname{Spt}_K \rightarrow \operatorname{Spt}_{K/H}$  is a right Quillen functor.

Since  $X \rightarrow X_{fG}$  is a trivial cofibration in  $\operatorname{Spt}_K$  and the map  $X \rightarrow X_{fK}$  has a fibrant target, there is a weak equivalence

$$\mu: X_{fG} \xrightarrow{\simeq} X_{fK}$$

in  $\operatorname{Spt}_K$ . Thus, there is a commutative diagram

$$\begin{array}{ccccc} ((X_{fG})^H)_{fK/H} & \xleftarrow[\simeq]{\eta} & (X_{fG})^H & \xrightarrow[\simeq]{\psi(X)_H^G} & ((X_{fG})_{fH})^H \\ & \searrow \widetilde{\mu^H} & \downarrow \mu^H & & \downarrow (\mu_{fH})^H \\ & & (X_{fK})^H & \xrightarrow{\psi(X)_H^K} & ((X_{fK})_{fH})^H. \end{array}$$

The map  $\widetilde{\mu^H}$  exists in  $\operatorname{Spt}_{K/H}$ , because in  $\operatorname{Spt}_K$  the map  $\eta$  is a trivial cofibration and  $(X_{fK})^H$  is fibrant. Also, as noted in the diagram, the map  $\psi(X)_H^G$  is a weak equivalence, since  $X$  is a hyperfibrant discrete  $G$ -spectrum, and the rightmost vertical map,  $(\mu_{fH})^H = \mu^{hH}$ , is a weak equivalence, since  $\mu$  is a weak equivalence in  $\operatorname{Spt}_H$ .

Now suppose that  $\psi(X)_H^K$  is a weak equivalence. Then  $\mu^H$  is a weak equivalence, and hence,  $\widetilde{\mu^H}$  is a weak equivalence between fibrant objects in  $\operatorname{Spt}_{K/H}$ . Therefore, there is a weak equivalence

$$\left(\widetilde{\mu^H}\right)^{K/H}: ((X_{fG})^H)^{hK/H} = (((X_{fG})^H)_{fK/H})^{K/H} \xrightarrow{\simeq} ((X_{fK})^H)^{K/H} = X^{hK},$$

completing the proof.  $\square$

## 8. DELTA-DISCRETE HOMOTOPY FIXED POINTS ARE ALWAYS DISCRETE HOMOTOPY FIXED POINTS

Let  $G$  be any profinite group. In this final section, we obtain in two different ways the conclusion stated in the section title. Somewhat interestingly, the discrete homotopy fixed points, in both cases, are those of a discrete  $G$ -spectrum that, in general, need not be equivalent to the delta-discrete  $G$ -spectrum (whose delta-discrete homotopy fixed points are under consideration).

Let  $\operatorname{holim}_\Delta X^\bullet$  be any delta-discrete  $G$ -spectrum and define

$$C(X^\bullet) := \operatorname{colim}_{N \triangleleft_o G} \left( \operatorname{holim}_{[n] \in \Delta} (X^n)_{f_G} \right)^N.$$

Notice that  $C(X^\bullet)$  is a discrete  $G$ -spectrum and

$$(\operatorname{holim}_\Delta X^\bullet)^{h_\delta G} \cong (\operatorname{holim}_{[n] \in \Delta} (X^n)_{f_G})^G \cong (C(X^\bullet))^G \xrightarrow{\simeq} (C(X^\bullet))^{hG},$$

where the justification for the second isomorphism above is as in [4, proof of Theorem 2.3] and the weak equivalence is due to the fact that the spectrum  $C(X^\bullet)$  is a fibrant discrete  $G$ -spectrum (by [4, Corollary 2.4]). Also, notice that the weak equivalence

$$(8.1) \quad (\operatorname{holim}_\Delta X^\bullet)^{h_\delta G} \xrightarrow{\simeq} (C(X^\bullet))^{hG},$$

defined above, is partly induced by the canonical map  $\iota_G$  (defined by a colimit of inclusions of fixed points) in the zigzag

$$(8.2) \quad C(X^\bullet) \xrightarrow{\iota_G} \operatorname{holim}_{[n] \in \Delta} (X^n)_{f_G} \xleftarrow{\simeq} \operatorname{holim}_\Delta X^\bullet$$

of  $G$ -equivariant maps. In (8.2), the second map, as indicated, is a weak equivalence of delta-discrete  $G$ -spectra.

Though (8.1) shows that the delta-discrete homotopy fixed points of any delta-discrete  $G$ -spectrum can be realized as the discrete homotopy fixed points of a discrete  $G$ -spectrum, there is a slight incongruity here: the map  $\iota_G$  in (8.2) does not have to be a weak equivalence. For example, if  $\iota_G$  is a weak equivalence, then  $\pi_0(\operatorname{holim}_\Delta X^\bullet)$  is a discrete  $G$ -module (since  $\pi_0(C(X^\bullet))$  is a discrete  $G$ -module), but this is not the case, for example, when  $\operatorname{holim}_\Delta X^\bullet$  is the delta-discrete  $\mathbb{Z}_q$ -spectrum  $Y^{h\mathbb{Z}/p}$  (this characterization of  $Y^{h\mathbb{Z}/p}$  is obtained by applying Lemma 2.8) that was referred to in the Introduction, since  $\pi_0(Y^{h\mathbb{Z}/p})$  is not a discrete  $\mathbb{Z}_q$ -module (by [5, Appendix A]).

Now we explain a second and more interesting way to realize  $(\operatorname{holim}_\Delta X^\bullet)^{h_\delta G}$  as a discrete homotopy fixed point spectrum. As in Section 3, let  $(X^\bullet)_{f_{ib}}$  denote the fibrant replacement of  $X^\bullet$  in the model category  $c(\operatorname{Spt}_G)$ . Then there is again a zigzag

$$(8.3) \quad \lim_\Delta (X^\bullet)_{f_{ib}} \xrightarrow{\lambda} \operatorname{holim}_\Delta (X^\bullet)_{f_{ib}} \xleftarrow[\gamma]{\simeq} \operatorname{holim}_\Delta X^\bullet$$

of canonical  $G$ -equivariant maps, where  $\lambda$  is the usual map in  $\operatorname{Spt}$  from the limit to the homotopy limit and the map  $\gamma$  is a weak equivalence of delta-discrete  $G$ -spectra. We will show that, as in the case of zigzag (8.2),  $\lim_\Delta (X^\bullet)_{f_{ib}}$  is a discrete  $G$ -spectrum and its discrete homotopy fixed points are equivalent to  $(\operatorname{holim}_\Delta X^\bullet)^{h_\delta G}$ , but, as before, the map  $\lambda$  need not be a weak equivalence.

**Lemma 8.4.** Let  $\operatorname{holim}_\Delta X^\bullet$  be a delta-discrete  $G$ -spectrum. Then  $\lim_\Delta (X^\bullet)_{f_{ib}}$  is a discrete  $G$ -spectrum.

*Proof.* Given two morphisms  $Y_0 \rightrightarrows Y_1$  in  $\operatorname{Spt}_G$ , let  $\operatorname{equal}[Y_0 \rightrightarrows Y_1]$  denote the equalizer in  $\operatorname{Spt}$ . Also, let  $\operatorname{equal}_G[Y_0 \rightrightarrows Y_1]$  denote the equalizer in  $\operatorname{Spt}_G$ . Due to the fact that  $X^\bullet$  and  $(X^\bullet)_{f_{ib}}$  are cosimplicial discrete  $G$ -spectra, to prove the lemma it suffices to show that  $\lim_\Delta X^\bullet$  is a discrete  $G$ -spectrum.

Since limits in  $\mathbf{Spt}$  are formed levelwise in simplicial sets and, given a cosimplicial simplicial set  $Z^\bullet$ , there is a natural isomorphism  $\lim_{\Delta} Z^\bullet \xrightarrow{\cong} Z^{-1}$ , where  $Z^{-1}$  is the equalizer of the coface maps  $Z^0 \xrightleftharpoons[d^1]{d^0} Z^1$  (see, for example, [23, Lemma 1]), it follows that the canonical  $G$ -equivariant map

$$\lim_{\Delta} X^\bullet \xrightarrow{\cong} \text{equal} \left[ X^0 \xrightleftharpoons[d^1]{d^0} X^1 \right]$$

is an isomorphism in  $\mathbf{Spt}$ .

The proof is completed by noting that there are isomorphisms

$$\begin{aligned} \text{equal}_G \left[ X^0 \xrightleftharpoons[d^1]{d^0} X^1 \right] &\cong \text{colim}_{N \triangleleft_o G} \left( \text{equal} \left[ X^0 \xrightleftharpoons[d^1]{d^0} X^1 \right] \right)^N \\ &\cong \text{equal} \left[ \text{colim}_{N \triangleleft_o G} (X^0)^N \xrightleftharpoons[d^1]{d^0} \text{colim}_{N \triangleleft_o G} (X^1)^N \right] \\ &\cong \text{equal} \left[ X^0 \xrightleftharpoons[d^1]{d^0} X^1 \right] \\ &\cong \lim_{\Delta} X^\bullet, \end{aligned}$$

with each isomorphism  $G$ -equivariant. In this string of isomorphisms, the first one is because of [3, Remark 4.2] and the third one follows from the fact that  $X^0$  and  $X^1$  are discrete  $G$ -spectra.  $\square$

**Lemma 8.5.** Let  $\text{holim}_{\Delta} X^\bullet$  be a delta-discrete  $G$ -spectrum. Then there is an equivalence

$$(\lim_{\Delta} (X^\bullet)_{\mathcal{F}ib})^{hG} \simeq (\text{holim}_{\Delta} X^\bullet)^{h_\delta G}.$$

*Proof.* Let  $\lim_{\Delta}^G Y^\bullet$  denote the limit in  $\mathbf{Spt}_G$  of an object  $Y^\bullet \in c(\mathbf{Spt}_G)$ . Since  $\lim_{\Delta} (X^\bullet)_{\mathcal{F}ib} \in \mathbf{Spt}_G$ , there are isomorphisms

$$(8.6) \quad \lim_{\Delta} (X^\bullet)_{\mathcal{F}ib} \cong \text{colim}_{N \triangleleft_o G} \left( \lim_{\Delta} (X^\bullet)_{\mathcal{F}ib} \right)^N \cong \lim_{\Delta}^G (X^\bullet)_{\mathcal{F}ib}$$

in  $\mathbf{Spt}_G$ . Since the functor  $\mathbf{Spt}_G \rightarrow c(\mathbf{Spt}_G)$  that sends a discrete  $G$ -spectrum  $Y$  to  $\text{cc}^\bullet(Y)$  (equipped with its natural  $G$ -action) preserves weak equivalences and cofibrations, its right adjoint  $\lim_{\Delta}^G (-): c(\mathbf{Spt}_G) \rightarrow \mathbf{Spt}_G$  preserves fibrant objects, and hence,  $\lim_{\Delta}^G (X^\bullet)_{\mathcal{F}ib}$  is a fibrant discrete  $G$ -spectrum, and hence, the canonical map

$$(\lim_{\Delta}^G (X^\bullet)_{\mathcal{F}ib})^G \xrightarrow{\simeq} (\lim_{\Delta}^G (X^\bullet)_{\mathcal{F}ib})^{hG} \cong (\lim_{\Delta} (X^\bullet)_{\mathcal{F}ib})^{hG}$$

is a weak equivalence.

The proof is completed by the zigzag of equivalences

$$(\lim_{\Delta}^G (X^\bullet)_{\mathcal{F}ib})^G \cong \lim_{\Delta} ((X^\bullet)_{\mathcal{F}ib})^G \xrightarrow{\simeq} \text{holim}_{\Delta} ((X^\bullet)_{\mathcal{F}ib})^G \xleftarrow{\simeq} (\text{holim}_{\Delta} X^\bullet)^{h_\delta G},$$

where the isomorphism uses (8.6), the first weak equivalence is by Theorem 3.2, and the second weak equivalence applies Theorem 3.4.  $\square$

The argument that was used earlier to show that  $\iota_G$  in zigzag (8.2) does not have to be a weak equivalence also applies to show that the map  $\lambda$  in zigzag (8.3) does not have to be a weak equivalence.

**Remark 8.7.** Notice that if  $(X^\bullet)_{fib}$  is fibrant in  $c(\text{Spt})$  (equipped with the injective model structure), then, as in the proof of Theorem 3.2, the map  $\lambda$  is a weak equivalence. Thus, interestingly, fibrations in  $c(\text{Spt}_G)$  are not necessarily fibrations in  $c(\text{Spt})$  – even though any fibration in  $\text{Spt}_G$  is a fibration in  $\text{Spt}$ , by [3, Lemma 3.10]. This observation is closely related to the fact that the forgetful functor  $U: \text{Spt}_G \rightarrow \text{Spt}$  need not be a right adjoint (see [1, Section 3.6] for a discussion of this fact): if one supposes that  $U$  is a right adjoint, then  $U$  is also a right Quillen functor, since  $U$  preserves fibrations and weak equivalences, and hence, by [24, Remark A.2.8.6], the forgetful functor  $U \circ (-): c(\text{Spt}_G) \rightarrow c(\text{Spt})$  preserves fibrations.

Let  $V$  be an open subgroup of  $G$ . By [5, proof of Lemma 3.1], the right adjoint  $\text{Res}_G^V: \text{Spt}_G \rightarrow \text{Spt}_V$  that regards a discrete  $G$ -spectrum as a discrete  $V$ -spectrum is a right Quillen functor, and hence (again, by [24, Remark A.2.8.6]), the restriction functor  $\text{Res}_G^V \circ (-): c(\text{Spt}_G) \rightarrow c(\text{Spt}_V)$  preserves fibrations. Thus,

$$(X^\bullet)_{fib} \cong \text{colim}_{N \triangleleft_o G} ((X^\bullet)_{fib})^N$$

is the filtered colimit of cosimplicial spectra  $((X^\bullet)_{fib})^N$ , each of which is fibrant in  $c(\text{Spt})$ , since  $\text{Res}_G^N \circ (X^\bullet)_{fib} = (X^\bullet)_{fib}$  is fibrant in  $c(\text{Spt}_N)$  and, as in the proof of Theorem 3.2, the functor  $(-)^N: c(\text{Spt}_N) \rightarrow c(\text{Spt})$  preserves fibrant objects. Since there are cases where  $(X^\bullet)_{fib}$  is not fibrant in  $c(\text{Spt})$ , we can conclude, perhaps somewhat surprisingly, that, unlike in  $\text{Spt}$ , a filtered colimit of fibrant objects in  $c(\text{Spt})$  does not have to be fibrant. This also shows that, though  $c(\text{Spt})$  is cofibrantly generated (by [24, Proposition A.2.8.2]), it is not weakly finitely generated (see [10, Definition 3.4, Lemma 3.5]).

A priori, we do not expect a delta-discrete  $G$ -spectrum, in general, to be equivalent to a discrete  $G$ -spectrum, and the fact that each of two natural ways to realize an arbitrary delta-discrete homotopy fixed point spectrum as a discrete homotopy fixed point spectrum fails to come from such a general equivalence is consistent with our expectation.

The following result shows that zigzags (8.2) and (8.3) are, in fact, directly related to each other (beyond just having the “same general structure”).

**Theorem 8.8.** Let  $\text{holim}_\Delta X^\bullet$  be any delta-discrete  $G$ -spectrum. Then the map  $\iota_G$  in (8.2) is a weak equivalence if and only if the map  $\lambda$  in (8.3) is a weak equivalence.

*Proof.* Let  $\text{holim}_\Delta^G Y^\bullet$  denote the homotopy limit in  $\text{Spt}_G$  of  $Y^\bullet$ , a cosimplicial discrete  $G$ -spectrum. Then, by [4, Theorem 2.3] and because homotopy limits are ends and thereby only unique up to isomorphism, there is the identity

$$\text{holim}_\Delta^G Y^\bullet = \text{colim}_{N \triangleleft_o G} (\text{holim}_\Delta Y^\bullet)^N,$$

which implies that  $C(X^\bullet) = \text{holim}_{[n] \in \Delta}^G (X^n)_{fG}$ . Similarly, there is the identity

$$\lim_\Delta^G (X^\bullet)_{fib} = \text{colim}_{N \triangleleft_o G} (\lim_\Delta (X^\bullet)_{fib})^N.$$

Also, since the map  $\{X^n\}_{[n] \in \Delta} \rightarrow \{(X^n)_{fG}\}_{[n] \in \Delta}$  is a trivial cofibration in  $c(\text{Spt}_G)$ , there is a weak equivalence

$$\tilde{\gamma}: \{(X^n)_{fG}\}_{[n] \in \Delta} \rightarrow (X^\bullet)_{fG}$$

in  $c(\text{Spt}_G)$ .

Given the above facts, there is the commutative diagram

$$\begin{array}{ccccc} \lim_{\Delta} (X^\bullet)_{fG} & \xrightarrow{\lambda} & \text{holim}_{\Delta} (X^\bullet)_{fG} & \xleftarrow[\widehat{\tilde{\gamma}}]{\simeq} & \text{holim}_{[n] \in \Delta} (X^n)_{fG} \\ \uparrow \cong & & \uparrow & & \uparrow \iota_G \\ \lim_{\Delta}^G (X^\bullet)_{fG} & \xrightarrow[\lambda'_G]{\simeq} & \text{holim}_{\Delta}^G (X^\bullet)_{fG} & \xleftarrow[\widehat{\tilde{\gamma}_G}]{\simeq} & \text{holim}_{[n] \in \Delta}^G (X^n)_{fG} \end{array}$$

of canonical maps: each vertical map is induced by inclusions of fixed points; the map  $\widehat{\tilde{\gamma}}$  is a weak equivalence (of delta-discrete  $G$ -spectra) because, in  $c(\text{Spt})$ ,  $\tilde{\gamma}$  is an objectwise weak equivalence between objectwise fibrant diagrams; the leftmost vertical map is an isomorphism thanks to Lemma 8.4; the proof of Lemma 8.10 below implies that  $\lambda'_G$  is a weak equivalence; and, in  $c(\text{Spt}_G)$ , the map  $\tilde{\gamma}$  is an objectwise weak equivalence between objectwise fibrant diagrams, so that, by [19, Theorem 18.5.3, (2)],  $\widehat{\tilde{\gamma}_G}$  is a weak equivalence.

The desired conclusion follows immediately from this diagram.  $\square$

Lemma 8.10 below, whose justification is used in the proof of the previous result, is an example of the statement that if  $\mathcal{C}$  is a small category and  $\mathcal{M}$  is a combinatorial simplicial model category, with  $M$  a fibrant object in the category  $\mathcal{M}^{\mathcal{C}}$  of  $\mathcal{C}$ -shaped diagrams in  $\mathcal{M}$ , when  $\mathcal{M}^{\mathcal{C}}$  is equipped with the injective model structure, then the canonical morphism  $\lim_{\mathcal{C}} M \rightarrow \text{holim}_{\mathcal{C}} M$  (defined as in [19, Example 18.3.8, (2)]) is a weak equivalence in  $\mathcal{M}$ . However, since the author is not able to point to a place in the literature that has an explicit proof of this particular statement, we give the details for the case that we need.

**Remark 8.9.** The author would like to qualify his comment related to the literature on homotopy limits that is in the last sentence above. Let  $\mathcal{N}$  be a simplicial model category such that, for some small category  $\mathcal{D}$ , the diagram category  $\mathcal{N}^{\mathcal{D}}$  has an injective model structure, with  $N$  fibrant in  $\mathcal{N}^{\mathcal{D}}$ . Then the statement that “the canonical morphism  $\lim_{\mathcal{D}} N \rightarrow \text{holim}_{\mathcal{D}} N$  is a weak equivalence in  $\mathcal{N}$ ” should follow in an essentially formal way from [19, Corollary 18.4.2, (1)] as its “model category dual,” but verifying this requires some care and the author has not checked all the details (when doing this, an application of [19, Corollary 18.4.5, (2)] is helpful).

Also, to place the above statement in a more general and useful framework, see [13]. In the dual case of the homotopy colimit, [30, Sections 5-9, 13] is helpful for relating [13] to the dual of the above statement. A helpful perspective is given in the discussion of homotopy right Kan extensions and homotopy limits in [24, Section A.2.8; e.g., Remark A.2.8.8].

**Lemma 8.10.** Let  $G$  be any profinite group and  $\mathcal{C}$  any small category. If  $Y^\bullet$  is fibrant in  $(\text{Spt}_G)^{\mathcal{C}}$ , when  $(\text{Spt}_G)^{\mathcal{C}}$  is equipped with the injective model structure, then the canonical map

$$\lambda_G: \lim_{\mathcal{C}}^G Y^\bullet \xrightarrow{\simeq} \text{holim}_{\mathcal{C}}^G Y^\bullet$$

is a weak equivalence in  $\text{Spt}_G$ . (Here, like before,  $\lim_{\mathcal{C}}^G Y^\bullet$  and  $\text{holim}_{\mathcal{C}}^G Y^\bullet$  denote the limit and homotopy limit, respectively, in  $\text{Spt}_G$  of  $Y^\bullet$ .)

*Proof.* By the initial comments in the proof of Theorem 8.8, we can identify the map  $\lambda_G$  with the canonical morphism

$$\lambda'_G: \text{colim}_{N \triangleleft_o G} (\lim_{C \in \mathcal{C}} Y^C)^N \rightarrow \text{colim}_{N \triangleleft_o G} (\text{holim}_{C \in \mathcal{C}} Y^C)^N$$

in  $\text{Spt}_G$ , and thus, to obtain the desired conclusion, it suffices to show that the map

$$\lambda''_G: \text{colim}_{N \triangleleft_o G} \lim_{C \in \mathcal{C}} (Y^C)^N \rightarrow \text{colim}_{N \triangleleft_o G} \text{holim}_{C \in \mathcal{C}} (Y^C)^N$$

is a weak equivalence in  $\text{Spt}$ .

As in Remark 8.7,  $\{(Y^C)^N\}_{C \in \mathcal{C}}$  is fibrant in  $\text{Spt}^{\mathcal{C}}$ , for each  $N$ , so that each  $\lim_{C \in \mathcal{C}} (Y^C)^N$  is fibrant in  $\text{Spt}$  and every  $(Y^C)^N$  is fibrant in  $\text{Spt}$ . This last conclusion implies that each  $\text{holim}_{C \in \mathcal{C}} (Y^C)^N$  is a fibrant spectrum. Therefore,  $\lambda''_G$  is the filtered colimit of the maps

$$\lambda^N: \lim_{C \in \mathcal{C}} (Y^C)^N \rightarrow \text{holim}_{C \in \mathcal{C}} (Y^C)^N,$$

each of which is a map between fibrant spectra, and hence, to show that  $\lambda''_G$  is a weak equivalence, we only have to show that each  $\lambda^N$  is a weak equivalence. Since  $\{(Y^C)^N\}_{C \in \mathcal{C}}$  is fibrant in  $\text{Spt}^{\mathcal{C}}$ , the main argument in the proof of Theorem 3.2 implies that  $\lambda^N$  is a weak equivalence, completing the proof.  $\square$

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